A new approach to the parametrization of the Cabibbo-Kobayashi-Maskawa matrix

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Abstract

The CKM-matrix V is written as a linear combination of the unit matrix I and a matrix U which causes intergenerational-mixing. It is shown that such a V results from a class of quark-mass matrices. The matrix U has to be hermitian and unitary and therefore can depend at most on 4 real parameters. The available data on the CKM-matrix including CP-violation can be reproduced by $V = (I+iU)/\sqrt{2}$. This is also true for the special case when U depends on *only 2 real parameters*. There is no CP-violating phase in this parametrization. Also, for such a V the invariant phase $\Phi \equiv \phi_{12} + \phi_{23} - \phi_{13}$, satisfies a criterion suggested for 'maximal' CP-violation.

It is more than twenty-five years since the first explicit parametrization for the six quark case was given [1] for the so called Cabibbo-Kobayashi-Maskawa (CKM) matrix. Since then many diferent parametrizations have been suggested [2, 3]. In this note, we wish to suggest a new approach to parametrizing the unitary CKM matrix V. For this purpose, we write V as a linear combination of the unit matrix I and another matrix U, so that

$$V(\theta) = \cos\theta I + i\sin\theta\, U \tag{1}$$

It is clear that for V to be unitary, U has to be both hermitian and unitary. Here θ is a parameter which will be fixed later. In Eq. (1), for the first term the physical (or the quark mass-eigenstate) and the gauge bases

are the same. The second term, through U, represents the difference in the two bases. It also causes inter-generational mixing and makes it possible for V to give CP-violating processes. The break-up of V in two parts makes it possible to have a simple parametrization. We now show that knowing $V(\theta)$ allows us to construct the quark-mass matrices in terms of the parameters of V and the quark-masses.

Form of the quark-mass matrices. In the gauge-basis, the part of the standard model Lagrangian relevant for us can be written as

$$\mathcal{L} = \bar{q}'_{uL} M_u q'_{uR} + \bar{q}'_{dL} M_d q'_{dR} + \frac{g}{\sqrt{2}} \bar{q}'_{uL} \gamma_\mu q'_{dL} W^+_\mu + H.c. \tag{2}$$

where $q'_u = (u', c', t')$ and $q'_d = (d', s', b')$. By suitable redefinition of the right-handed quark fields one can make the quark-mass matrices M_u and M_d hermitian. Let the diagonal forms of the hermitian M_u and M_d be given by

$$\hat{M}_u = V_u^\dagger M_u V_u, \qquad \hat{M}_d = V_d^\dagger M_d V_d. \tag{3}$$

In the physical basis, defined by $q_\alpha = V_\alpha^\dagger q'_\alpha$ ($\alpha = u$ or d), one has

$$\mathcal{L} = \sum_{\alpha=u,d} \bar{q}_{\alpha L} \hat{M}_\alpha q_{\alpha R} + \frac{g}{\sqrt{2}} \bar{q}_{uL} \gamma_\mu V q_{dL} W^+_\mu + H.c. \tag{4}$$

where

$$V = V_u^\dagger V_d \tag{5}$$

is the CKM-matrix.

For a V given by Eq.(1), one can easily find V_u and V_d which satisfy Eq. (5) In general,

$$V_u = V(\theta_u) = \cos\theta_u I - i \sin\theta_u U \tag{6}$$

$$V_d = V(\theta_d) = \cos\theta_d I + i \sin\theta_d U \tag{7}$$

will give $V(\theta)$ provided $\theta_u + \theta_d = \theta$. This is so since $V(\theta_1)V(\theta_2) = V(\theta_1 + \theta_2)$ because $U = U^\dagger$ and $U^2 = I$.

Given these V_u and V_d, Eq.(3) then *determines* M_u and M_d in terms of the quark masses and the experimentally accessible parameters of the CKM-matrix. More formally, this means that in the spectral decomposition of $M_u(M_d)$ the projectors depend only on the parameters in $V(\theta)$ and $\theta_u(\theta_d)$. There is a freedom in the choice of the values θ_u and θ_d as only their sum $\theta_u + \theta_d = \theta$ is determined from knowing $V(\theta)$.

It is clear that our form of $V(\theta)$ provides an explicit solution for a class of quark mass matrices.

Form of U in the standard model. To determine the general form of the hermitian and unitary 3×3 matrix U we start with a general hermitian matrix

$$U = \begin{pmatrix} u_1 & \alpha^* & \beta^* \\ \alpha & u_2 & \gamma^* \\ \beta & \gamma & u_3 \end{pmatrix} \tag{8}$$

where u_i $(i=1,2,3)$ are real and α, β and γ are complex numbers. Requiring U to be unitary as well implies that $U^2 = I$. Explicitly this gives

$$u_1^2 + |\alpha|^2 + |\beta|^2 = 1, \tag{9}$$

$$u_2^2 + |\alpha|^2 + |\gamma|^2 = 1, \tag{10}$$

$$u_3^2 + |\beta|^2 + |\gamma|^2 = 1; \tag{11}$$

and

$$|\alpha|\,(u_1+u_2) + |\beta\gamma|\exp(i\phi) = 0, \tag{12}$$

$$|\beta|\,(u_1+u_3) + |\alpha\gamma|\exp(-i\phi) = 0, \tag{13}$$

$$|\gamma|\,(u_2+u_3) + |\alpha\beta|\exp(i\phi) = 0. \tag{14}$$

Here $\phi \equiv \phi_\alpha - \phi_\beta + \phi_\gamma$ while ϕ_α, ϕ_β and ϕ_γ are the phases of α, β and γ. Eqs. (12-14) immediatly imply that $\sin\phi = 0$ or $\phi = 0$ or π. The resulting U in the two cases differ by an overall sign [4]. For definiteness we consider the case $\phi = 0$. Eqs. (12-14) determine the diagonal elements in terms of $|\alpha|$, $|\beta|$ and $|\gamma|$ and substituting these in Eqs. (9-11) gives the constraint

$$\left|\frac{\alpha\beta}{\gamma}\right| + \left|\frac{\alpha\gamma}{\beta}\right| + \left|\frac{\beta\gamma}{\alpha}\right| = 2. \tag{15}$$

Using this one has

$$u_1 = \left|\tfrac{\alpha\beta}{\gamma}\right| - 1, \quad u_2 = \left|\tfrac{\alpha\gamma}{\beta}\right| - 1 \quad \text{and } u_3 = \left|\tfrac{\beta\gamma}{\alpha}\right| - 1. \tag{16}$$

For a more convenient form of U, we put

$$\alpha = -2bc^*, \quad \beta = -2ac, \quad \text{and } \gamma = -2a^*b. \tag{17}$$

Since, $\phi_\alpha = (\phi_b - \phi_c) + \pi$ etc., the condition $\phi = 0$ translates into

$$\phi_a - \phi_b + \phi_c = \frac{\pi}{2}, \tag{18}$$

where ϕ_a, ϕ_b and ϕ_c are the phases of the complex numbers a, b and c. The constraint of Eq. (15) becomes

$$|a|^2 + |b|^2 + |c|^2 = 1. \tag{19}$$

The general expression of the hermitian and unitary U in terms of a, b and c is

$$U = I - 2 \begin{pmatrix} |a|^2 + |b|^2 & b^* c & a^* c^* \\ bc^* & |a|^2 + |c|^2 & ab^* \\ ac & a^* b & |b|^2 + |c|^2 \end{pmatrix} \tag{20}$$

Given the two constraints in Eq. (18) and Eq. (19), we note that a general hermitian and unitary 3×3 matrix depends on at most four real parameters. This is the form of U we will use [4].

The Jarslskog invariant [5] for U, viz. $J(U) = Im(U_{11}U_{22}U_{12}^*U_{21}^*) = 0$. However, the $V(\theta)$ in Eq. (1) does give CP-violation, since

$$J(V(\theta)) = 8\cos\theta \sin^3\theta |abc|^2 = \cos\theta |V_{12}||V_{13}||V_{23}| \tag{21}$$

In our case, there is no 'CP-violating phase' which governs the finitess of J. One of the off-diagonal elements of $V(\theta)$ has to be zero for J to vanish. Note, that J is just given in terms of $|V_{ij}|(i \neq j)$ unlike usual parametrizations [3]. It is interesting to note, that even when a,b,c are pure imaginary [6] so that $V(\theta)$ depends on only 3 real parameters, $J(V(\theta))$ is non-zero. In this case, U becomes real and symmetric and the only complex number in $V(\theta)$ is i in Eq.(1) !

Since U is hermitian it requires that $|V_{ij}| = |V_{ji}|$ for $V(\theta)$ in Eq. (1). The experimentally determined CKM-matrix V_{EX} given by the Particle Data Group [3]

$$V_{EX} = \begin{pmatrix} 0.9745 - 0.9760 & 0.2170 - 0.2240 & 0.0018 - 0.0045 \\ 0.2170 - 0.2240 & 0.9737 - 0.9753 & 0.0360 - 0.0420 \\ 0.0040 - 0.0130 & 0.0350 - 0.0420 & 0.9991 - 0.9994 \end{pmatrix} \tag{22}$$

The entries correspond to the ranges for the moduli of the matrix elements. It is clear that $|V_{12}| = |V_{21}|$ and $|V_{23}| = |V_{32}|$ are satisfied for the whole range, while the equality $|V_{13}| = |V_{31}|$ is suggested by the data. Given the fact that $|V_{13}|$ and $|V_{31}|$ are the hardest to determine experimentally, it is possible they might turn out to be equal. We adopt a common numerical value viz.

$|V_{13}| = |V_{31}| = 0.005825 \pm 0.002925$. This numerical value is obtained by first converting the range of values in V_{EX} into a central value with errors, so that $|V_{13}| = 0.00315 \pm 0.00135$ and $|V_{31}| = 0.0085 \pm 0.0045$. The average of these two gives the common numerical value above. Ranges for other moduli also are converted into a central value with errors.

To confront $V(\theta)$ with experiment we need to specify θ. A physically appealing choice is to give equal weight to the generation mixing term (U) and the generation diagonal term (I) in $V(\theta)$, so that $\theta = \pi/4$ and the CKM-matrix

$$V(\pi/4) = \frac{1}{\sqrt{2}}(I + iU). \tag{23}$$

We use this for numerical work.

Numerical results Experimentaly, $|V_{12}|$ and $|V_{23}|$ are well determined. We take their average (or central) value in the range given in Eq. (22) as inputs; that is, $|V_{12}| = |V_{21}| = 0.2205$ and $|V_{23}| = |V_{32}| = 0.039$. Given these, one has

$$|a| = |V_{23}|/(2\sin\theta|b|), \tag{24}$$

$$|c| = |V_{12}|/(2\sin\theta|b|). \tag{25}$$

The constraint Eq. (19), gives a quadratic equation for $|b|^2$ with the solutions,

$$|b|^2 = \frac{1}{2}\left[1 \pm \sqrt{1 - (|V_{12}|^2 + |V_{23}|^2)\csc^2\theta}\right]. \tag{26}$$

Note, for real $|b|^2$, above input implies $\sin^2(\theta) \geq 0.05014$ or $\theta \geq 12.94°$. Since, $|V_{12}| > |V_{23}| > |V_{13}|$ it is clear we need the positive sign in Eq. (26) so that $|b| > |c| > |a|$. For $\theta = \pi/4$, Eqs. (24-26) yield,

$$|a| = 0.02794, \qquad |b| = 0.98705, \qquad |c| = 0.15796. \tag{27}$$

The values of the $|V_{ij}|$ for $V(\pi/4)$ in Eq. (23) are given in Table I. The values in the table should be compared with the average values of $|V_{ij}|$ obtained from V_{EX}. For example, average of V_{11} from Eq. (22) is $\frac{1}{2}(0.9745 + 0.9760) = 0.97525$. This is given as 0.97525 ± 0.00075. The 'error' indicates the range for $|V_{11}|$. The experimental $|V_{ij}|$ are given in column 2, while the calculated values are given in column 3. The agreement is quite satisfactory

suggesting that a CKM-matrix with$|V_{ij}| = |V_{ji}|$ may fit the data. We did not attempt a best fit in view of our assumption $|V_{13}| = |V_{31}|$.

The value of J for V_{EX} and $V(\pi/4)$ are also given in the Table. $J(V_{EX})$ was calculated using the formula [7]

$$J^2 = |V_{11}V_{22}V_{12}V_{21}|^2 - \frac{1}{4}[1-|V_{11}|^2-|V_{22}|^2-|V_{12}|^2-|V_{21}|^2+|V_{11}V_{22}|^2+|V_{12}V_{21}|^2]^2 \tag{28}$$

with the central values of $|V_{ij}|$, $i = 1, 2$ and since these four are best measured. The value $J(V(\pi/4))$ was calculated using Eq. (21) and is about $3-4$ times smaller. This is reasonable considering the slight differences in values of $|V_{i,j}|$ $i = 1, 2$ in the two cases and also since there is a strong numerical cancellation between the two terms on the r.h.s of Eq. (28).

It is important to note that calculated values require only the knowledge of $|a|$, $|b|$ and $|c|$. Thus, the numerical results are valid even when a, b and c are pure imaginary [6] and $V(\pi/4)$ depends on only 2 real parameters [8].

Concluding remarks Apart, form providing a good numerical fit with 4 or possibly 2 parameters, the CKM-matrix $V(\pi/4)$ has an interesting feature connected with a criterion [9] for 'maximal' CP-violation.

It was noted [9] that physically the relevant phase for CP-violation in the CKM-matrix V is $\Phi = \phi_{12} + \phi_{23} - \phi_{13}$, where ϕ_{ij} is the phase of the matrix element V_{ij}. The reason for this is because Φ is invariant under rephasing transformations of V. So, a value of $\Phi \equiv |\pi/2|$ was suggested as corresponding to 'maximal' CP-violation. This is so in our case because of the constraint in Eq. (18) since $\Phi = 2(\phi_a + \phi_c - \phi_b) - \pi/2$. So, $\cos\Phi = 0$ for $V(\pi/4)$. Note that, $\Phi = \pi/2$ is automatic when a, b and c are pure imaginary [6] and in that case $V(\pi/4)$ depends on only 2 real parameters.

It is remarkable that $V(\pi/4)$ with only 2 real parameters fits the available data. This may be because only the absolute values $|V_{ij}|$ are known at present. Future information on the full V_{ij} will tell us if the relations [10] implied by the two parameter parametrization given here are viable or the more general four parameter parametrization would be needed. It would be very interesting if the symmetry relations $|V_{ij}| = |V_{ji}|(i \neq j)$ are confirmed experimentally.

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References

[1] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).

[2] L. Maiani, Phys. Lett.**62B**, 183 (1976); L Wolfenstein, Phys. Rev.Lett. **51**, 1945 (1983); L.-L. Chau and W.-Y. Keung, Phys. Rev. Lett.**53**, 1802 (1984); H. Harari and M. Leurer, Phys. Lett. **B181**, 123 (1986); H. Fritzsch and J. Plankl, Phys. Rev. **D35**, 1732 (1987); P. Kielanowski, Phys. Rev. Lett. **63**, 2189 (1989); A. Mondragon and E. Rodriguez-Jáuregui, Phys. Rev. **D59**, 093009-1 (1999).

[3] Particle Data Group, C. Caso et al., The European Physical Journal, **C3**, 1, (1998), and references therein.

[4] In case of the choice $\phi = \pi$, u_i have the opposite sign, so that $u_1 = 1 - \left|\frac{\alpha\beta}{\gamma}\right|$ etc. Then puting $\alpha = +2bc^*$ etc. gives $\phi_a - \phi_b + \phi_c = -\frac{\pi}{2}$ and the resulting matrix is just -1 times U given in Eq. (20). Note in either case, $\cos(\phi_a - \phi_b + \phi_c) = 0$.

[5] C. Jarlskog, Phys. Rev. Lett.**55**, 1039 (1985).

[6] Note that for pure imaginary a, b and c, one has $\phi_a = \phi_b = \phi_c = \pi/2$. So Eq. (18) is automatically satisfied. In this case, U and $V(\pi/4)$ depend on only two real parameters.

[7] C. Hamzaoui, Phys. Rev. Lett.**61**, 35 (1988).

[8] For a two parameter fit with a different approach see paper by P. Kielanowski cited in ref. 1.

[9] M. Gronau and J. Schechter, Phys. Rev. Lett.**54**, 385 (1985), M. Gronau, R. Johnson and J. Schechter, Phys. Rev. Lett.**54**, 2176 (1985). Other references can be found in these.

[10] For example $|V_{ij}| = |V_{ji}| (i \neq j)$. Also phases ϕ_{ij} are fixed and other relations betwen absolute magnitudes exist. For $V(\pi/4)$, an interesting relation (implied by Eq.(19) and Eq.(18)) is $2J = \sqrt{2}|V_{12}V_{13}V_{23}| = |V_{12}V_{23}|^2 + |V_{12}V_{13}|^2 + |V_{23}V_{13}|^2$

Quantity	Experiment	Theory
$\|V_{12}\| = V_{21}$	0.2205 ± 0.0035	0.2205 (input)
$\|V_{23}\| = V_{32}$	0.0390 ± 0.0030	0.039 (input)
$\|V_{13}\| = V_{31}$	0.005825 ± 0.002925	0.00624
V_{11}	0.97525 ± 0.00075	0.975367
V_{22}	0.9745 ± 0.0008	0.974607
V_{33}	0.99925 ± 0.00015	0.99922
J	1.414×10^{-4}	3.795×10^{-5}

Table I. Numerical values of the moduli of the matrix elements of $V(\theta)$ for $\theta = \pi/4$. Experimental values are average values obtained from V_{EX} in Eq. (13). The 'errors' reflect the large of values for $|V_{ij}|$. Note, since $|V_{13}| = 0.00315 \pm 0.00135$ and $|V_{31}| = 0.0085 \pm 0.0045$, we grote the average of these in the Table. J is the Jarslskog invariant (see text)